Solutions to the exercises in Lecture 1

In Exercises 2–9, A is an abelian ℓ-group and $\leq$ is the partial order on A induced by $\vee$, as in Exercise 1.

1. Let $\langle L, \vee \rangle$ be a join semilattice. Define a relation $\leq$ on L by

$$a \leq b \iff: b = a \vee b.$$

   Show that $\leq$ is a partial order, and for all $a, b \in L$, $a \vee b$ is the least upper bound of a and b.

   Solution. Suppose $a, b, c \in L$.

   - Reflexivity. $a \leq a$ because $a = a \vee a$ by idempotence of $\vee$.
   - Anti-symmetry. Suppose $a \leq b$ and $b \leq a$. Then $b = a \vee b$ and $a = b \vee a$. By commutativilty of $\vee$, $a = b$.
   - Transitivity. Suppose $a \leq b$ and $b \leq c$. Then $b = a \vee b$ and $c = b \vee c$. Thus, $c = (a \vee b) \vee c = a \vee (b \vee c) = a \vee c$, by associativity of $\vee$. Thus, $a \leq c$.

   This shows that $\leq$ is a partial order. We show that $a \vee b$ is the least upper bound of a and b. It follows from the definition of $\leq$ and the idempotence and associativity of $\vee$ that $a \leq a \vee b$ and $b \leq a \vee b$, i.e., $a \vee b$ is an upper bound of a and b. Suppose c is an upper bound of a and b, i.e., $c = a \vee c$ and $c = b \vee c$. Then $c = c \vee c = (a \vee c) \vee (b \vee c) = (a \vee b) \vee c$ by the properties of $\vee$. It follows that $(a \vee b) \leq c$. Thus $a \vee b$ is the least upper bound of a and b.

2. Show that for all $a, b, c, d \in A$: if $a \leq b$ and $c \leq d$ then $a + c \leq b + d$. (In particular, $a \leq b \iff 0 \leq b - a$.)

   Solution. If $a \leq b$, then $b = a \vee b$, so $b + c = (a \vee b) + c = (a + c) \vee (b + c)$, so $a + c \leq b + c$. If $c \leq d$, then by a similar argument, $b + c \leq b + d$. By transitivity of $\leq$, $a + c \leq b + d$.

3. *Consequences of distributivity of* $+$ *over* $\vee$. Show that for all $a, b \in A$:

   (i) $a = a^+ - a^-$.
   $a + (0 \vee -a) = (a \vee 0)$, so $a = (a \vee 0) - (0 \vee -a)$

(ii) $(a+b)^+ \le a^+ + b^+$.
$(a+b) \vee 0 \le (a+b) \vee a \vee b \vee 0 = \big(a + (b \vee 0)\big) \vee (b \vee 0) = (a \vee 0) + (b \vee 0)$.

(iii) $(b-a)^+ \ge b^+ - a^+ \ge -\big((a-b)^+\big)$.
Writing $b-a$ in place of b in the previous fact: $a^+ + (b-a)^+ \ge b^+$, proving the first inequality. Similarly $(a-b)^+ \ge a^+ - b^+$, so $b^+ - a^+ \ge -\big((a-b)^+\big)$.

(iv) If $a \wedge b = 0$, then $(a-b)^+ = a$ and $(a-b)^- = b$.
$a = a - (a \wedge b) = a + (-a \vee -b) = 0 \vee (a-b) = (a-b)^+$
$b = b - (a \wedge b) = b + (-a \vee -b) = 0 \vee (b-a) = (a-b)^-$

(v) For all $n \in \mathbb{N}$, $n\,(a \vee 0) = n\,a \vee (n-1)\,a \vee \cdots \vee a \vee 0$.
Proof by induction. The $n=0$ case is obvious. By the induction hypothesis:

$$(n+1)(a \vee 0) = (a \vee 0) + \big(na \vee (n-1)a \vee \cdots \vee 0\big).$$

The right side simplifies to

$$(n+1)a \vee na \vee (n-a)a \vee \cdots \vee 0.$$

(vi) From *(v)* deduce: if $0 < n \in \mathbb{N}$ and $0 \le n\,a$ then $0 \le a$.

$$\begin{aligned}
n\,(a \vee 0) &= n\,a \vee (n-1)\,a \vee \cdots \vee a \vee 0 && \text{Exercise } 3(v)\\
&= n\,a \vee (n-1)\,a \vee \cdots \vee a && n\,a \vee 0 = n\,a\\
&= a + \big((n-1)\,a \vee (n-2)\,a \vee \cdots \vee 0\big) && \text{distributive law}\\
&= a + (n-1)(a \vee 0) && \text{Exercise } 3(v)\\
a \vee 0 &= a && \text{Subtract } (n-1)(a \vee 0)
\end{aligned}$$

4. *Properties of* $|\ |$. Show that for all $a, b \in A$:

(i) $0 \le |a|$.
$a \le |a|$ and $-a \le |a|$. Adding these, $0 \le 2|a|$. By $3(vi)$, $0 \le |a|$.

A self-contained proof:

$$\begin{aligned}
(|a| \vee 0) + (|a| \vee 0) &= 2|a| \vee |a| \vee 0 && + \text{ distributes over } \vee\\
&= 2|a| \vee |a| && 0 \le 2|a|\\
&= |a| + (|a| \vee 0) && + \text{ distributes over } \vee
\end{aligned}$$

(*ii*) $a^+ + a^- = |a|$.
$a^+ + a^- = (a \vee 0) + (-a \vee 0) = \big(a + (-a \vee 0)\big) \vee (-a \vee 0) = a \vee -a \vee 0 = |a|$

(*iii*) $|a + b| \leq |a| + |b|$.
By 3(*ii*), $(a + b)^- = (-a - b)^+ \leq (-a)^+ + (-b)^+ = a^- + b^-$. Add this to 3(*ii*).

5. Show (*i*) for all $x, y, z \in A$, $x + (y \wedge z) = (x + y) \wedge (x + z)$, and (*ii*) $a \mapsto -a$ is an order-reversing automorphism of A. (Accordingly, $a \wedge b$ is the greatest lower bound of a and b, and therefore, $\langle A, \vee, \wedge \rangle$ is a lattice.)

   (*i*): $x + (y \wedge z) = x - (-y \vee -z) = -(-x + (-y \vee -z) = -\big((-x-y) \vee (-x-z)\big) = (x + y) \wedge (x + z)$. (*ii*): Suppose $a \leq b$. Then $b = a \vee b$. Subtract $a + b$ from both sides: $-a = -b \vee -a$. Thus $-b \leq -a$.

6. Suppose $a_i, b_j \in A$ and $a_i \wedge b_j = 0$ for $i = 1, \ldots m$ and $j = 1, \ldots n$. Show:

   (*i*) $(a_1 + a_2) \wedge b_1 = 0$.
   $a_1 = a_1 + (a_2 \wedge b_1) = (a_1 + a_2) \wedge (a_1 + b_1)$, so $0 = (a_1 + a_2) \wedge (a_1 + b_1) \wedge b_1$, but $(a_1 + b_1) \wedge b_1 = b_1$, since $0 \leq a_1$.

   (*ii*) $(a_1 + \cdots + a_m) \wedge (b_1 + \cdots + b_n) = 0$.
   By induction, using part (*i*).

7. Suppose $a, b \in A$ and $n \in \mathbb{N}$. Show:

   (*i*) $n(a^+) = (n\,a)^+$.
   $n\,a = n(a^+) - n(a^-)$. By 6(*ii*), $n(a^+) \wedge n(a^-) = 0$. Now use 3(*iv*).

   (*ii*) $n\,(a \vee b) = n\,a \vee n\,b$, and $n\,(a \wedge b) = n\,a \wedge n\,b$.
   $n\,(a \vee b) = n\,\big((a - b)^+ + b\big) = (n\,a - n\,b)^+ + n\,b = n\,a \vee n\,b$

8. Show that $\langle A, \vee, \wedge \rangle$ is a distributive lattice.

   In this case, the hint actually provides a pretty complete proof sketch. *Hint.* It is enough to show that $(x \wedge y)^+ = x^+ \wedge y^+$. The relation $(x \wedge y)^+ \leq x^+ \wedge y^+$ is immdiate. For the other inequality, let $z := (x \wedge y)^+ - (x \wedge y)$. Show $0 \leq x + z$ & $x \leq x + z$ and hence $x^+ \leq x + z$. Similarly $y^+ \leq y + z$. Thus $x^+ \wedge y^+ \leq (x \wedge y) + z = (x \wedge y)^+$.

9. Suppose $X \subseteq A$ is a set. Show that the set of all elements of A that can be written in the form

$$\bigvee_{i=1}^{p} \bigwedge_{j=1}^{q} \sum_{k=1}^{r} n_{ijk} x_{ijk},$$

where p, q, r, n_{ijk} are positive integers and $x_{ijk} \in X$, is closed under $-, +$ and $\vee$.

It suffices to show that the negative of an expression of this form can be written in this form, and that the sum and the sup of two such expressions can also be written in this form. In all cases, the distributive laws are enough to do this. The details, however, are complicated. See the Bigard-Keimel-Wolfenstein text.